Behaviour of Radon-Nikodym cocycles of one-ended measure class preserving transformation

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Overview

1. Preliminaries
2. Derivation of the cocycle
3. Example: Oscillating cocycles
4. General periodic measures
5. Example: Cocycle going to zero nonsummably

Preliminaries

Our main space of interest is the **Cantor space,** denoted $2^{\mathbb{N}}$, and we will only consider product measures $\mu = \prod_{n \in \mathbb{N}} \mu_n$ where μ_n is a probability measure on $\{0, 1\}$. We call two product measures μ, ν **equivalent**, denoted $\mu \sim$, if $\mu(A) = 0 \iff \nu(A) = 0$ for all measurable $A \subseteq 2^{\mathbb{N}}$.

A function $T : (2^{\mathbb{N}}, \mu) \to (2^{\mathbb{N}}, \mu)$ induces an equivalence relation, the **orbit equivalence relation**, denoted $\mathbb{E}_T$, whose classes are the orbits of T.

We call a function T **measure class preserving** (also null-preserving, quasi-pmp), if for any μ-measurable $A \subset 2^{\mathbb{N}}$,
$\mu(A) = 0 \iff \mu(T^{-1}(A) = 0)$. We moreover call μ $\mathbb{E}_T$-**quasi-invariant** if the saturation $[A]_{\mathbb{E}_T}$ is null whenever A is null.

Radon-Nikodym Cocycle

If μ is $\mathbb{E}_T$-quasi-invariant, then $\mathbb{E}_T$ admits an almost-everywhere unique **Radon-Nikoydm cocycle** $\mathfrak{w} : \mathbb{E}_T \to \mathbb{R}^+$. [KM04]

For μ-a.e $x \mathbb{E}_T y \mathbb{E}_T z$, we have $\mathfrak{w}^x(z) = \mathfrak{w}^x(y) \cdot \mathfrak{w}^y(z)$.

This cocycle measures the relative weights of points in equivalence classes, that is, for $x \mathbb{E}_T y$, $\mathfrak{w}^x(y)$ can be thought of the "weight" that x assigns to y. So, for a Borel set A, we have

$$\mu(T(A)) = \int_A \mathfrak{w}^x(T(x)) d\mu(x)$$

The Bit Flip Transformation

To define the function of interest, we restrict to the subset of $\mathcal{S} \subseteq 2^{\mathbb{N}}$ where every sequence has infinitely many 1s. We define the map $\tau : S \to S$ as the flip of the first 1 in a sequence to a 0. Note that τ induces the eventual equivalence $\mathbb{E}_0$ as an orbit equivalence relation. E.g.

$$\tau(010101...) = 000101...$$

$$\tau(10001...) = 00001...$$

τ has countably many inverse functions $(\tau_k^{-1})_{k\geq 1}$, where τ_k^{-1} flips the 0 at the k-th index to a 1.

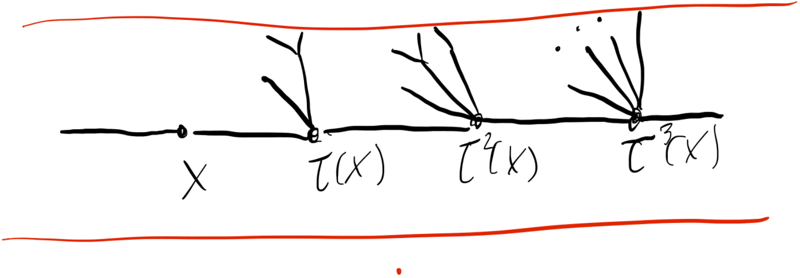
Graphing of the Bit Flip

$x = 01011\ldots$

$\tau(x) = 00011\ldots$

$\tau^2(x) = 00001\ldots$

$\tau^3(x) = 00000\ldots$



Measure Class Preserving?

We are interested in the Radon-Nikodym cocycle of τ, but we need to first find out which measures μ are $\mathbb{E}_0$-quasi-invariant.

For this, it is enough to check that τ and all of it's right inverses τ_k^{-1} are measure class preserving

Indeed, since τ flips one bit, in order to preserve null sets, we require that the measure of that bit is nontrivial. Thus we have that any product measure μ with nontrivial marginal measures μ_n are $\mathbb{E}_0$-quasi-invariant.

Kakutani's characterization

For other functions we can check they are measure class preserving by comparinng them to their pushforward measures using Kakutani's characterization of equivalent product measures [Kak48],

Theorem (Kakutani, 1948)

If μ, ν are product measures on $2^{\mathbb{N}}$ for which $\mu_n \sim \nu_n$ for each $n \in \mathbb{N}$. Then

$$\mu \sim \nu \iff \sum_{n \in \mathbb{N}} (\sqrt{\mu_n(0)} - \sqrt{\nu_n(0)})^2 + (\sqrt{\mu_n(1)} - \sqrt{\nu_n(1)})^2 < \infty$$

Computing the Cocycle

Using a convenient cylinder set, we compute the cocycle associated to the transformation, namely $A = [00\ldots1]_0^n$. Then we have

$$\frac{d\tau_*\mu}{d\mu}(A) = \frac{[00\ldots0]_0^n}{[00\ldots1]_0^n} = \frac{\mu_n(0)}{\mu_n(1)}.$$

So for any $k \in \mathbb{N}$,

$$\mathfrak{w}^x(\tau^k(x)) = \prod_{i=0}^{k} \frac{\mu_{n_i}(0)}{\mu_{n_i}(1)}$$

where the sequence $(n_i)_{i\geq1}$ is the positions of the ones in x.

Example of the Cocycle

Let $x := 010010001000\ldots$. Then x has 1s at indices $2, 5, 9$, etc., so

$$\mathfrak{w}(x, \tau(x)) = \frac{\mu_2(0)}{\mu_2(1)},$$

$$\mathfrak{w}(x, \tau^2(x)) = \frac{\mu_2(0)}{\mu_2(1)} \cdot \frac{\mu_5(0)}{\mu_5(1)},$$

and

$$\mathfrak{w}(x, \tau^3(x)) = \frac{\mu_2(0)}{\mu_2(1)} \cdot \frac{\mu_5(0)}{\mu_5(1)} \cdot \frac{\mu_9(0)}{\mu_9(1)}.$$

Question

What is

$$\lim_{k\to\infty} \mathfrak{w}^x(\tau^k(x))?$$

I.e. What is the behaviour of the cocycle as we travel towards the forward end of the graphing

Example 1: Constant Cocycle

If

$$\mu_n(0) = \mu_n(1) = \frac{1}{2}$$

for all n, then

$$\frac{\mu_n(0)}{\mu_n(1)} = 1$$

for all n, so

$$\mathfrak{w}(x, \tau^k(x)) = \prod_{i=0}^{k} \frac{\mu_{n_i}(0)}{\mu_{n_i}(1)} = 1$$

for all x, k.

Example 2: Vanishing Cocycle

If

$$\mu_n(0) = \frac{1}{3}, \mu_n(1) = \frac{2}{3}$$

for all n, then

$$\frac{\mu_n(0)}{\mu_n(1)} = \frac{1}{2}$$

for all n, so

$$\mathfrak{w}(x, \tau^k(x)) = \prod_{i=0}^{k} \frac{\mu_{n_i}(0)}{\mu_{n_i}(1)} = \frac{1}{2^k}$$

for all x, k, so

$$\lim_{k\to\infty} \mathfrak{w}(x, \tau^k(x)) = 0.$$

Example 3: Exploding Cocycle

If we tweak the measures from example 2, i.e.

$$\mu_n(0) = \frac{2}{3}, \mu_n(1) = \frac{1}{3}$$

for all n, then

$$\frac{\mu_n(0)}{\mu_n(1)} = 2$$

for all n, so

$$\mathfrak{w}(x, \tau^k(x)) = \prod_{i=0}^{k} \frac{\mu_{n_i}(0)}{\mu_{n_i}(1)} = 2^k$$

for all x, k, so

$$\lim_{k \to \infty} \mathfrak{w}(x, \tau^k(x)) = \infty.$$

Question

Does the cocycle ever oscillate?

Oscillating Cocycle

$$\mathfrak{w}^x(\tau^k(x)) = \prod_{i=0}^{k} \frac{\mu_{n_i}(0)}{\mu_{n_i}(1)}$$

We need the μ_ns to oscillate, in order to have some $\frac{\mu_n(0)}{\mu_n(1)}$ greater than 1, and some less than 1.

Since

$$\frac{\mu_n(0)}{\mu_n(1)} < 1 \leftrightarrow \mu_n(1) > \frac{1}{2},$$

the cocycle will be biased towards smaller terms.

First Attempt

For even n, let

$$\mu_n(0) = \frac{1}{3}, \mu_n(1) = \frac{2}{3}.$$

For odd n, let

$$\mu_n(0) = \frac{2}{3}, \mu_n(1) = \frac{1}{3}.$$

Then

$$\frac{\mu_n(0)}{\mu_n(1)} = \begin{cases} \frac{1}{2}, & n \text{ even} \\ 2, & n \text{ odd} \end{cases}$$

But we are twice as likely to have $x_n = 1$ when n is even, than when n is odd, so

$$\mathfrak{w}^x(\tau^k(x)) \to 0.$$

How can we correct this imbalance?

Second Attempt

Double the frequency of the large terms, since they are half as likely to be factored into the cocycle.

For $n = 0 \mod 3$,

$$\mu_n(0) = \frac{1}{3}, \mu_n(1) = \frac{2}{3}$$

For $n = 1, 2 \mod 3$,

$$\mu_n(0) = \frac{2}{3}, \mu_n(1) = \frac{1}{3}$$

This way we have a repeating pattern of measures, where there are more measures of the second form to account for the fact that 1s will appear with lesser probability in those positions.

Building a Random Walk 1/3

We want to see how the cocycle behaves for a "typical" $x \in 2^{\mathbb{N}}$, so we view $x = (x_n)_{n \geq 1} \sim \mu$ as a random variable.
Let P_k be the contribution of x_k to the cocycle:

$$P_k = \begin{cases} \frac{1}{2}, x_k = 1 \text{ and } k \equiv_3 0 \\ 2, x_k = 1 \text{ and } k \equiv_3 1, 2 \\ 1 \text{ if } x_k = 0 \end{cases}$$

Let

$$S_k := \sum_{i \leq k} \log_2 P_i, \text{ then } S_k \in \mathbb{Z}.$$

$$S_k - S_{k-1} \in \{-1, 0, 1\}.$$

Building a Random Walk 2/3

Then

$$S_k = \log \mathfrak{w}^{N_k}(x),$$

where

$$N_k := \sup\{n \leq k : x_n = 1\}.$$

Let

$$X_k := S_{3k} - S_{3(k-1)}.$$

Then

$$X_k \in \{-1, 0, 1, 2\}.$$

Let $a := x_{3k-2}x_{3k-1}x_{3k}$. Then

$$X_k = \begin{cases} 2 \text{ if } a = 110 \\ 1 \text{ if } a = 010, 100, \text{ or } 111 \\ 0 \text{ if } a = 000, 101, \text{ or } 011 \\ -1 \text{ if } a = 001 \end{cases}$$

Building a Random Walk 3/3

Since the marginal measures are periodic with period 3, the X_ks are IID, and

$$S_{3k} = \sum_{i=1}^{k} X_i.$$

So $(S_{3k})_{k \geq 1}$ is a random walk.

Chung-Fuchs Theorem

Now that the cocycle has the form of a random walk, we use the following theorem [CF51]:

Theorem (Chung, Fuchs 1951)

For a random walk $S_k = \sum_{i=1}^{k} X_i$ on $\mathbb{R}$ where X_i are iid and are bounded,

- $\mathbb{E}[X_k] = 0 \leftrightarrow \limsup_{k\to\infty} S_k = \infty$ *and* $\liminf_{k\to\infty} S_k = -\infty$ μ*-a.e.*
- $\mathbb{E}[X_k] < 0 \leftrightarrow \lim_{k\to\infty} S_k = -\infty$ μ*-a.e.*
- $\mathbb{E}[X_k] > 0 \leftrightarrow \lim_{k\to\infty} S_k = \infty$ μ*-a.e.*

Our X_ks have expectation 0, so the S_{3k}s will oscillate.
Hence, $\log \mathfrak{w}^k$ oscillates $\Rightarrow \mathfrak{w}^k$ oscillates.

Expectation computation

$$\begin{cases} -1: & x_{3k-2}x_{3k-1}x_{3k} = 001, & q_{-1} \\ 0: & 000, 101, 101, & q_0 \\ 1: & 010, 100, 111, & q_1 \\ 2: & 110, & q_2 \end{cases}$$

$$\mathbb{E}[X_k] = -q_{-1} + q_1 + 2q_2 = -\frac{8}{27} + \frac{6}{27} + 2\frac{1}{27} = 0$$

Example 1

For $n = 0 \mod 3$, let

$$\mu_n(0) = \frac{1}{3}, \mu_n(1) = \frac{2}{3}$$

and for $n = 1, 2 \mod 3$, let

$$\mu_n(0) = \frac{2}{3}, \mu_n(1) = \frac{1}{3}.$$

Then by the Chung-Fuchs Theorem, the cocycle $\mathfrak{w}^x(\tau^k(x))$ of μ will oscillate μ-a.e.

Questions

Will the cocycle oscillate when...

- μ is defined analogously to the above example, but with period > 3?
- μ is an arbitrary periodic measure?

Examples 2 to ∞

We can generalize the above example to obtain a family of periodic measures, all of which have oscillating cocycles.
For a fixed $m > 2$, define μ as follows:
If $n \equiv 0 \bmod m$, then

$$\mu_n(0) = \frac{1}{m}, \mu_n(1) = \frac{m-1}{m}$$

If $n \equiv 1, ..., m-1 \bmod m$, then

$$\mu_n(0) = \frac{m-1}{m}, \mu_n(1) = \frac{1}{m}$$

Examples 2 to ∞ cont'd

Let P_k be the contribution of x_k to the cocycle:

$$P_k = \begin{cases} \frac{1}{m-1}, x_k = 1 \text{ and } k \equiv_m 0 \\ m-1, x_k = 1 \text{ and } k \equiv_m 1, 2, ..., m-1 \\ 1 \text{ if } x_k = 0 \end{cases}$$

Let

$$S_k := \sum_{i \leq k} \log_{m-1} P_i, \text{ then } S_k \in \mathbb{Z}.$$

Examples 2 to ∞ cont'd

Let

$$X_k := S_{mk} - S_{m(k-1)}.$$

Then

$$X_k \in \{-1, 0, 1, ..., m-1\}.$$

Since the marginal measures are periodic with period m, the X_ks are IID, so $(S_{mk})_{k \geq 1}$ is a random walk.

We now want to compute

$$\mathbb{E}[X_k] = -q_{-1} + q_1 + 2q_2 + ... + (m-1)q_{m-1}.$$

First, notice that $X_k = -1$ if and only if

$$x_{mk} = 1 \text{ and } x_{mk+1} = ... = x_{mk+m-1} = 0,$$

so

$$q_{-1} = (\frac{m-1}{m})^m.$$

Continue to get:

$$q_i = \binom{m}{i+1}(\frac{1}{m})^{i+1}(\frac{m-1}{m})^{m-i-1}.$$

Examples 2 to ∞ cont'd

So

$$\mathbb{E}[X_k] = \sum_{j=0}^{m} (j-1) \binom{m}{j} (\frac{1}{m})^j (\frac{m-1}{m})^{m-j}$$

$$= \sum_{j=0}^{m} j \binom{m}{j} (\frac{1}{m})^j (\frac{m-1}{m})^{m-j} - \sum_{j=0}^{m} \binom{m}{j} (\frac{1}{m})^j (\frac{m-1}{m})^{m-j}$$

$$= 1 - (\frac{1}{m} + \frac{m-1}{m})^m = 0$$

by the binomial theorem.
So by the Chung-Fuchs Theorem, for all $m \geq 3$, the product measure μ will have a cocycle with the desired oscillatory behavior μ-a.e.

Classifying Cocycles of Periodic Measures

We can always classify the limit behavior of the cocycle of a periodic measure μ, using the Chung-Fuchs Theorem:

Theorem (Chung, Fuchs 1951)

For a random walk (S_k) on $\mathbb{R}$ with X_i that are iid and bounded,

- $\mathbb{E}[X_k] = 0 \leftrightarrow \limsup_{k\to\infty} S_k = \infty$ *and* $\liminf_{k\to\infty} S_k = -\infty$ μ*-a.e.*
- $\mathbb{E}[X_k] < 0 \leftrightarrow \lim_{k\to\infty} S_k = -\infty$ μ*-a.e.*
- $\mathbb{E}[X_k] > 0 \leftrightarrow \lim_{k\to\infty} S_k = \infty$ μ*-a.e.*

Classifying Cocycles of Periodic Measures cont'd

Let μ be an arbitrary measure with periodic marginals of period $T > 2$. We construct a random walk in a completely analogous way to the previous examples:

$$S_k := \sum_{i \leq k} \log P_i \text{ and } X_k := S_{Tk} - S_{T(k-1)}$$

for $k > 0$. Then

$$\lim_{k\to\infty} \mathfrak{w}^k(x) \begin{cases} \text{oscillates}, & \mathbb{E}[X_1] = 0 \\ = 0, & \mathbb{E}[X_1] < 0 \\ = \infty, & \mathbb{E}[X_1] > 0 \end{cases}$$

A nonsummable example

Another question was whether the cocycle ever goes to zero, but in a nonsummable way.

We construct a product measure where this will occur. To do so, first let (n_k) be a sequence defined as $n_0 = 0$ and $n_{k+1} = n_k + 2^{k \sum_{i \leq k} i}$.

We define our marginal measures as:

$$\begin{cases} n \in (n_k) : \mu_n(0) = \frac{1}{2^k+1},\ \mu_n(1) = \frac{2^k}{2^k+1} \\ n \notin (n_k) : \mu_n(0) = \mu_n(1) = \frac{1}{2} \end{cases}$$

Nonsummable cont.

Now, letting $m_k = \sum_{i \leq k} i$, we let

$$E_k := \{x \in 2^{\mathbb{N}} : \text{ x has at most } m_k \text{ 1s between indices } n_k \text{ and } n_{k+1}\}.$$

Notice that if the sequence $\mathfrak{w}^x(\tau(x))$ is summable, then $x \in E_k$ for infinitely many k.

We can show that $\mu(E_k) \leq \frac{1}{2^k}$, so by Borel-Cantelli, the set of all x whose cocycle is summable lies in a null set, hence the cocycle is nonsummable almost everywhere.

Bound calculation

$$\mu(E_k) = \frac{1}{2^{2^{km_k}}} \sum_{i=0}^{2^{m_k}} \binom{2^{km_k}}{i} \leq \frac{1}{2^{2^{km_k}}} \sum_{i=1}^{2^{m_k}} \frac{2^{ikm_k}}{i!}$$

$$\leq \frac{1}{2^{2^{km_k}}} 2^{m_k} 2^{2^{m_k} km_k} = 2^{m_k + 2^{m_k} km_k - 2^{km_k}}$$

$$\leq 2^{-k}$$

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